ON OPERATOR THEORETICAL INTERPRETATION FOR SOME CLASSICAL PROBLEMS IN GEOMETRY AND DIFFERENTIAL EQUATIONS

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ABSTRACT. A consequence of the Gauss Bonnet theorem is interpreted in term of operator theory by Alain Connes in his book, Non Commutative geometry. In this note we explain in details about his method. We also introduce an operator theoretical nature for limit cycle theory.

INTRODUCTION

The stability of the integral of Gaussian curvature of a surface under small deformation is proved in the book of Alain Connes, see [1, page 13] . His approach is essentially based on the stability of value $\tau(E, E, E)$ under the one parameter deformation of projection E in a C^* algebra where τ is a 3_cyclic cocycle. In this paper we explain in detail his proof.

Aside of the Gauss Bonnet theorem, we consider an important object in the theory of ordinary differential equations, so called "limit cycles". A limit cycle is an isolated closed orbit for planar vector field

$$(1) \qquad x' = P(x, y), \;\; y' = Q(x, y)$$

This vector field sometimes is denoted by $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$.

Limit cycles are the main objects of the second part of the Hilbert 16th problem which asks:

Hilbert 16th problem. Does there exist a uniform upper bound $H(n)$, depending only on n, for the number of limit cycles of (1), where $P(x, y)$ and $Q(x, y)$ are real polynomials of degree n?

The vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ gives us a linear operator D on the space $C^\infty(\mathbb{R}^2)$, the space of all complex valued smooth functions on $\mathbb{R}^2$ as follows:

$$(2) \qquad D(U) = PU_x + QU_y$$

Note that the space of Schwartz functions $\mathcal{S}$ is invariant under the operator D if P and Q are polynomial functions. Recall that $\mathcal{S}$ is the the subspace of $C^\infty(\mathbb{R}^2)$ consists of all function f which all partial derivatives tends to zero at infinity.

We observe that the codimension of the rang of (2) is an upper bound for the number of limit cycles of (1). This observation is true in both case that domain of D is either $C^\infty(\mathbb{R}^2)$ or the Schwartz functions $\mathcal{S}$. So it is very important to know whether D is a fredholm operator or a semi fredholm operator. Among all differential operators, the elliptic operators are the most well behaved operators, from the view of fredholm theory. But (2) defines a first order differential operator $D$with real coefficient which obviously is not an elliptic operator. In fact an n_th

order differential operators on $\mathbb{R}^k$ with real coefficients is not elliptic if n is an odd integer.

So apparently the operator (2) is useless, but we have a remedy for this problem. We associate to D an nth order partial differential operator $\widetilde{D}$ with polynomial coefficient such that $\widetilde{D}$ and D are similar operators, as two linear maps on the space $\mathcal{S}$ of Schwartz functions. The advantage of $\widetilde{D}$ is that it has the chance to be an elliptic operator when n is an even number while the first order operator D is never elliptic. The similarity of D and $\widetilde{D}$ obviously implies that codimension of rang D is equal to the codimension of rang $\widetilde{D}$. On the other hand the codimension of rang D is an upper bound for the number of limit cycles of (2).

We have two reason for that we restrict (2) to Schwartz functions. The first reason is that we loos the similarity of D and $\widetilde{D}$ on whole space $C^\infty(\mathbb{R}^2)$. The second reason is that the Schwartz functions are very important subspace of Sobolov spaces which are the main place to apply fredholm index theory. Fortunately, as we said above, the Schwartz space $\mathcal{S}$ is invariant under (2) when P and Q are polynomials functions.

The Gauss Bonet theorem and operator theory

Let S be a compact surface in $\mathbb{R}^3$. The classical Gauss Bonnet theorem says that

$$(3) \qquad \iint_S \mathcal{K}dS = \mathcal{X}(S)$$

where $\mathcal{K}$ is the gaussian curvature of S and $\mathcal{X}$ is the Euler characteristic of S. This is an amazing identity because the left side is a geometric quantity but the right side is a topological one. The topology of surface S do not change with an small deformation. So $\iint_S \mathcal{K}dS$ is stable under small deformation. In this section we explain in detail the cyclic cocycles interpretation for this stability as described in [1, page 13]

Let $N = (f, g, h)$ be the Gauss normal map as a map $N : S \to S^2$. Then the following formula can be found in any book on differential geometry of surfaces:

$$(4) \qquad \iint_S \mathcal{K}dS = \sum_i \iint_{G_i} Det \begin{pmatrix} N & N_u & N_v \end{pmatrix} dudv = \sum_i \iint_{G_i} Det \begin{pmatrix} f & f_u & f_v \\ g & g_u & g_v \\ h & h_u & h_v \end{pmatrix}$$

where $G_i's \subseteq \mathbb{R}^2$ are the domain of parameterizations of S.

Note that $df = f_u du + f_v dv$, $dg = g_u du + g_v dv$ and $dh = h_u du + h_v dv$. By expanding the determinant in (4) in term of first column we have

$$\iint_S \mathcal{K}dS = \sum_i \iint_{G_i} fdgdh + \sum_i \iint_{Gi} gdhdf + \sum_i \iint_{Gi} hdfdg$$

We replace the variables u and v in $\mathbb{R}^2$ with points of surface S we obtain:

$$\iint_S \mathcal{K}dS = \iint_S fdgdh + \iint_S gdhdf + \iint_S hdfdg$$

Consider the differential 1_forms $\alpha = fgdh$ and $\beta = ghdf$. Since S is a two dimensional manifold without boundary then the Stocks theorem implies that $\iint_S d(\alpha) = \iint_S d(\beta) = 0$. This shows that

$$(5) \qquad \iint_S fdgdh = \iint_S gdhdf = \iint_S hdfdg$$

So we have

$$(6) \qquad \iint_S \mathcal{K}dS = 3 \iint_S fdgdh$$

The above equation is a motivation to define a trilinear form τ on the algebra $C^\infty(S)$ with $\tau(f_0, f_1, f_2) = \iint_S f_0 df_1 df_2$. Then (5) shows that

(7) $$\tau(f_0, f_1, f_2) = \tau(f_1, f_2, f_0)$$

Furthermore a simple application of Leibnitz formula implies that

$$f_0 f_1 d(f_2) d(f_3) - f_0 d(f_1 f_2) d(f_3) + f_0 d(f_1) d(f_2 f_3) - f_3 f_0 d(f_1 d(f_2)) = 0$$

so we have

(8) $$\tau(f_0 f_1, f_2, f_3) - \tau(f_0, f_1 f_2, f_3) + \tau(f_0, f_1, f_2 f_3) - \tau(f_3 f_0, f_1, f_2) = 0$$

The relation (7) is a cyclic property and the relation (8) is a cocycle property. That is every τ which satisfies in (8) is a cocycle of a complex with respect to the Hochschild coboundary map. So we have the following definition

Definition 1. A trilinear form on an algebra A is called a 3_cyclic cocycle if it satisfies in (7) and (8).

Let a be an element of a unital algebra A and $i, j \in \{1, 2\}$. By $\delta_{ij}(a)$ we mean a 2 by 2 matrices in $M_2(A)$ which has only one nonzero entry a at i, th row and j, th column. Every 2 by 2 matric can be written as a sum of the above special matrices. Now assume that τ is a 3_cyclic cocyle on A. We extend τ on $M_2(A)$ by

$$\tau(\delta_{ij}(a), \delta_{i'j'}(a'), \delta_{i''j''}(a'')) = \tau(a, a', a'') trace\,(\delta_{ij}(1) \delta_{i'j'}(1) \delta_{i''j''}(1))$$

where "trace" is the standard trace on $M_2(\mathbb{C})$.
Now consider the 3_cyclic cocycle $\tau(f, g, h) = \iint_S f dg dh$ on the algebra $A = C^\infty(S)$. A long but straightforward computation shows that the extension τ on $M_2(A)$ satisfies $\tau(E, E, E) = \lambda \iint_S f dg dh$ where λ is an independent constant and E has the following form

(9) $$E = 1/2 \begin{pmatrix} 1 - h & f + ig \\ f - ig & 1 - h \end{pmatrix}$$

Here f, g and h are in $C^\infty(S)$ with $f^2 + g^2 + h^2 = 1$. We observe that

(10) $$E = E^2 = E^*$$

where $*$ is the natural involution of $M_2(C^\infty(S))$. An element of a C^* algebra which satisfies (10) is called a projection. Now considering (6) we have the following formula for the integral of the gaussian curvature on a surface S with the Gauss normal map $N = (f, g, h)$:

(11) $$\iint_S \mathcal{K} = \lambda \tau(E, E, E)$$

where E is as in (9) and λ is a universal constant.

Now assume that S and S' are two surfaces with the Gauss normal maps $N = (f, g, h)$ and $N' = (f', g', h')$ and Gaussian curvatures $\mathcal{K}$ and $\mathcal{K}'$, respectively. We denote by E and E', the corresponding projections of S and S' as defined in (9). Let $\phi : S \to S'$ be a diffeomorphism between two surfaces. Put $\widetilde{f} = f' \circ \phi$, $\widetilde{g} = g' \circ \phi$ and $\widetilde{h} = h' \circ \phi$ then as a consequence of the integral version of the change of variable formula we have $\iint_S \widetilde{f} d\widetilde{g} d\widetilde{h} = \iint_{S'} f' dg' dh' = \iint_{S'} \mathcal{K}' = \tau(E', E', E')$. Suppose that S' is a result of an small deformation of S. This implies that S' is diffeomorphic to S and N' is an small deformation of N. In particular E' is an small perturbation of E. In the situation of small perturbation, the following proposition says that $\tau(E, E, E) = \tau(E', E', E')$. Thus

$$\iint_S \mathcal{K} = \iint_{S'} \mathcal{K}'$$

This equality shows the stability of the integral of curvature of surfaces under deformations.

Before we state the proposition, we recall the homotopy equivalent and unitary equivalent of projections in a C^* algebra. Two projection E and E' are homotopy equivalent if there is a continuous curve $E(t)$ of projections, $t \in [0,\ 1]$, such that $E(0) = E$, $E(1) = E'$. E is unitary equivalent to E' if there is a unitary element u such that $E' = uEu*$. If $E(t)$ is continuous curve of projections, then there is a continuous curve of unitaries $u(t)$ with $u(0) = 1$ such that

(12) $$E = E(t) = u(t)E(0)u(t)^*$$

   see [5, Corollary 5.2.9]

Proposition 1. Let $E(t)$ be a differentiable curve of projections in a C^* algebra A. Suppose that τ is a 3_cyclic cocycle on A. Then $\tau(E(t), E(t), E(t))$ is a constant complex number, for all t.

Proof. Let $E(t)$ be a differentiable curve of projections with $E(0) = E$. Then

$$\dot{E} = dE/dt = \lim_{t \to 0} \frac{E(t) - E}{t}$$
$$= \lim_{t \to 0} \frac{u(t)Eu(t)^* - E}{t}$$
$$= \lim_{t \to 0} \frac{u(t)E - Eu(t)}{t} u^*(t)$$
$$= \lim_{t \to 0} \frac{u(t)E - Eu(t)}{t}$$

This shows that $\dot{E}$ is in the form of a commutator. That is $\dot{E} = EX - XE$ for a continuous curve $X(t)$ in the algebra A. Now the time derivative of trilinear form $\tau(E, E, E)$ is equal to $\tau(\dot{E}, E, E) + \tau(E, \dot{E}, E) + \tau(E, E, \dot{E}) = 0$. Because we substitute $\dot{E}$ by $XE - EX$ and use (7) and (8). Thus $\tau(E, E, E)$ is a constant scalar.   □

The number of limit cycles via differential operator theory

Let A be a subalgebra of $C^\infty(\mathbb{R}^2)$. We say that A separates compact submanifolds of $\mathbb{R}^2$ if for every finite number of disjoint smooth closed curves $\gamma_1, \ldots, \gamma_n$ there is a real valued $f \in A$ such that $f(x_i) \neq f(x_j)$ for all $x_i \in \gamma_i$ and all $x_j \in \gamma_j$, $i \neq j$. Obviously $C^\infty(\mathbb{R}^2)$ or algebra $\mathcal{S}$ of Schwartz function can separates smooth closed curves in the plane. Let X be a smooth vector field as in (1), in the introduction. This vector field defines the first order differential operator D, as in (2). Then we have:

Proposition 3. Let A be a self adjoint subalgebra of $C^\infty(\mathbb{R}^2)$ which separates compact submanifolds of $\mathbb{R}^2$. Assume that A is invariant under the the differential operator D in (2). Then the number of closed orbits of (1) is less than or equal to the codimension of the rang of D restricted to A.

Proof. Let $\gamma(t) = (x(t), y(t))$ be a periodic solution for (1). Then for every real valued $g \in \mathbb{C}^\infty(\mathbb{R}^2)$ we have $D(g)(\gamma(t)) = \frac{d(g \circ \gamma)}{dt}(\gamma(t))$. Since $g \circ \gamma$ is a periodic function, its time derivative must be vanished at some time $t_0 \in \mathbb{R}$. So $D(g)$ must vanished on at least one point of each periodic orbit γ. Let (1) has n periodic solutions(closed orbits) $\gamma_1, \ldots, \gamma_n$. Assume that $f \in A$ be a real valued function which separates γ_i's. Suppose that for real numbers $\lambda_0, \ldots, \lambda_{n-1}$, $\sum_{i=0}^{n-1} \lambda_i f^i$ is in the rang of D. We show that $\lambda_i = 0$ for $i = 0, 1, \ldots, n-1$. Consider the polynomial

$p(z) = \sum_{i=0}^{n-1} \lambda_i z^i$. If p is a nonzero polynomial, then there exist an $i \in \{1, 2, \ldots, n\}$ such that $p(z) \neq 0$ for all $z \in \gamma_i$, since γ_i's are n disjoint closed curves. This shows that there is no real valued function $g \in A$ such that $D(g) = p(f)$. since A is self adjoint and the differential operator D has real coefficients, we conclude that every nontrivial combination $\sum_{i=0}^{n-1} \lambda_i f^i$ with complex coefficients is not in the rang of D, since f is a real valued function. This shows that the dimension of A/rang of D, as a complex vector space, is at least n $\qquad\square$

Remark 2. We observe that the codimension of the rang of the differential operator D play an important role in counting the number of closed orbits of a vector field. The codimension of the range of an operator is closely related to the concept of fredholm index. Recall that the fredholm index of an operator D is equal to $\dim \ker D - \dim Coker D$ where $\dim coker D$ is the codimension of the rang of D. On the other hand, the most relevant differential operators which can be a fredholm operator are elliptic operators. An elliptic operator is a a differential operator for which its principle part satisfies in certain nondependence condition, for definition of elliptic operators see [4]. Now the problem is that a differential operator D of first order (of odd order) can not be an elliptic operator, if the coefficients of D are real valued functions.

Now assume that the $P(x, y)$ and $Q(x, y)$ in (2) are polynomials of degree n. Then the space of Schwartz functions $\mathcal{S}$ is invariant under D in (2). Let $\mathcal{F}$ be the Fourier transform which is a bijection from $\mathcal{S}$ to $\mathcal{S}$. Then obviously two operators D and $\widetilde{D} = \mathcal{F}^{-1} D \mathcal{F}$ are similar operators, hence their range have equal codimensions. Moreover $\widetilde{D}$ is an n_th order differential operator with polynomial coefficient. There are examples which shows that $\widetilde{D}$ can be an elliptic operator.

Example . Let $P(x, y)$ and $Q(x, y)$ in (1) be two polynomials of degree two which last homogeneous part are $P_2(x, y) = ax^2 + bxy + cy^2$ and $Q_2(x, y) = dx^2 + exy + fy^2$. Then using the standard formula for the Fourier transform, which can be found for example in [4, page 154], we obtain that the principle part of the differential operator $\mathcal{F}^{-1} D \mathcal{F}$ is equal to

$$(ax + dy)U_{xx} + (bx + ey)U_{xy} + (cx + fy)U_{yy}$$

Now by a very simple computation we observe that this second order operator is an elliptic operator for some appropriate coefficient of P_2 and Q_2.

Remark 3. As we see in the above example, in some special case ,we can obtain an elliptic operator from a polynomial vector field on the plane. But in reality, the elliptic operators are fredholm operators only on compact manifolds, while $\mathbb{R}^2$ is not compact. On the other hand, there is an standard method, so called the Poincare compactification which carries a polynomial vector fields to an analytic vector field on S^2. For more information on poincare compactification see [3]. Now it would be interesting to carry the corresponding elliptic operator $\widetilde{D} = \mathcal{F}^{-1} D \mathcal{F}$ from $\mathbb{R}^2$ to S^2. What smooth vector bundle on S^2 is the best framework for consideration of such (possible) elliptic operator?

We end the paper with the following question:

Question . The linear operator (2) can be considered as a pure algebraic linear map on $\mathbb{R}[x, y]$, the polynomial ring in two variables, when P and Q are polynomial. Is there an example of such operator such that the codimension of the range of D is a finite number different from 0 and 1. Note that there are trivial example of codimension equal to 0, 1 and ∞. These codimension occur for the following vector field, respectively: $\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x}$.

References

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